Math 246B Lecture 13 Notes

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1 Mittag-Leffler's Theorem and Infinite Products of Holomorphic Functions

1.1 Mittag-Leffler's theorem

Last time, we showed that if $\Omega \subseteq \mathbb{C}$ is open and $f \in C^1(\Omega)$, then there exists some $u \in C^1(\Omega)$ such that $\frac{\partial u}{\partial \bar{z}} = f$ in Ω. Here is an application.

Theorem 1.1 (Mittag-Leffler)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A \subseteq \Omega$ be a set with no limit points in Ω. For each $a \in A$, let p_a be a rational function of the form*

$$p_a(z) = \sum_{j=1}^{N_a} \frac{c_{a_j}}{(z-a)^j}$$

for some c_{a_j} where $1 \leq N_a < \infty$. Then there exists a $f \in \mathrm{Hol}(\Omega \setminus A)$ such that for all $a \in A$, $f - p_a$ is holomorphic in a neighborhood of a.

Remark 1.1. In other words, f is a meromorphic function in Ω with poles only in A, and for any $a \in A$, p_a is the singular part of the Laurent expansion of f at a.

Proof. The idea is to solve the problem first in the smooth (C^1) category and then correct a smooth solution to get a holomorphic solution solving a $\overline{\partial}$-problem.

The set A is at most countable, and we may assume A is infinite: $A = \{a_1, a_2, \dots\}$. Let $U_j \subseteq \Omega$ be a small neighborhood of a_j such that $\overline{U}_j \cap \overline{U}_\ell = \varnothing$ for $j \neq \ell$, and let $\varphi_j \in C_0^k(U_j)$, where $k \geq 2$, be such that $\varphi_j = 1$ in a neighborhood of a_j. Define

$$g(z) = \sum_{j=1}^{\infty} p_{a-j}(z) \varphi_j(z)$$

for $z \in \Omega \setminus A$. For every compact $K \subseteq \Omega$, $U_j \cap K = \varnothing$ for all but finitely many j. So $g \in C^k(\Omega \setminus A)$, and near a_j, $g - p_{a_j} \equiv 0 \in C^K$.

Next, compute

$$\frac{\partial g}{\partial \overline{z}} = \sum_{j=1}^{\infty} \frac{\partial}{\partial \overline{z}}(p_{a_j}\varphi_j) = \sum_{j=1}^{\infty} p_{a_j}\frac{\partial \varphi_j}{\partial \overline{z}},$$

which is 0 near a_j for any j. Since $\frac{\partial g}{\partial \overline{z}} = 0$ on A, $\frac{\partial g}{\partial \overline{z}}$ extends to a C^{k-1} function on Ω: $\frac{\partial g}{\partial \overline{z}} \in C^{k-1}(\Omega) \subseteq C^1(\Omega)$. Now let $u \in C^1(\Omega)$ be such that $\frac{\partial u}{\partial \overline{z}} = \frac{\partial g}{\partial \overline{z}}$ in Ω. Define $f(z) = g(z) - u(z) \in C^1(\Omega \setminus A)$. Then $\overline{\partial} f = 0$, so $f \in \mathrm{Hol}(\Omega \setminus A)$. In a neighborhood of $a_j \in A$, we write

$$f - p_{a_j} = \underbrace{g - p_{a_j}}_{\in C^k \text{ near } a_j} - \underbrace{u}_{\in C^1}.$$

Then $f - p_{a-j}$ is bounded in a set of the form $0 < |z - a_j| < r_j$ for small r_j, so $f - p_j$ has a removable singularity at a_j. So $f - p_{a_j}$ is holomorphic near a_j for all j. $\square$

1.2 Infinite products of holomorphic functions

Next, we will discuss Weierstrass's theorem, which basically says that any subset of $\Omega \subseteq \mathbb{C}$ with no limit points in Ω is the zero set of some holomorphic function. The idea is to try infinite products of holomorphic functions. You can see how Mittag-Leffler's theorem is inspired by this result.[1]

Proposition 1.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let (f_j) be a sequence in $\mathrm{Hol}(\Omega)$. Assume that for every compact $K \subseteq \Omega$, there exists $N \in \mathbb{N}$ and a convergence series $\sum_{j=N}^{\infty} M_j < \infty$ with $M_j \geq 0$ such that f_j is nonvanishing on K for all $j \geq N$ and that $|\operatorname{Log}(f_j(z))| \leq M_j$, wher $j \geq N$, and $z \in K$. This is the principal branch of $\log$: $\arg \in (-\pi, \pi]$. Then the sequence $(\prod_{j=1}^{n} f_j)$ converges locally uniformly in Ω, $f(z) := \lim_{n\to\infty} \prod_{j=1}^{n} f_j(z) \in \mathrm{Hol}(\Omega)$, and we write $f(z) = \prod_{j=1}^{\infty} f_j(z)$. The zeros of f are given by the union of the zeros of the f_j, counting multiplicities.*

Proof. Let $K \subseteq \Omega$ be compact, and let N, M_j be as in the proposition. For $j \geq N$, write $f_j = e^{\operatorname{Log}(f_j)}$. Then

$$\prod_{j=N}^{n} f_j = \exp \underbrace{\left(\sum_{j=N}^{n} \operatorname{Log}(f_j)\right)}_{\text{converges uniformly on } K},$$

so, using $|e^z - e^w| \leq e^{\max(\operatorname{Re}(z), \operatorname{Re}(w))}|z - w|$, we write

$$\left|\prod_{j=N}^{n} - \prod_{j=N}^{m} f_j\right| \leq C_K \sum_{j=n+1}^{m} |\operatorname{Log}(f_j)| \to 0$$

[1] Mittag-Leffler was a student of Weierstrass.

uniformly on K. To show that $|e^z - e^w| \leq e^{\max(\mathrm{Re}(z),\mathrm{Re}(w))}|z-w|$, note that

$$e^z - e^w = \int_0^1 \frac{f}{dt} e^{tz+(1-t)w}\, dt. \qquad \square$$

Example 1.1. Assume that $(f_j) \in \mathrm{Hol}(\Omega)$ is such that for every compact $K \subseteq \Omega$, we have $\sum_{j=1}^{\infty} \sup_K |1 - f_j| < \infty$ (normal convergence on each compact). Then the proposition applies, and the product $\prod_{j=1}^{\infty} f_j$ converges locally uniformly in Ω.